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THE AREA INTEGRAL ASSOCIATED TO SELF-ADJOINT OPERATORS ON WEIGHTED MORREY SPACES

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Abstract

This paper is concerned with some norm inequalities for the area integrals associated to a non-negative self-adjoint operator satisfying a pointwise Gaussian estimate for its heat kernel, on weighted Morrey spaces.

1. Introduction

The classical Morrey spaces were introduced by Morrey in [17] to investigate the local behaviour of solutions to second order elliptic partial

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differential equations. The boundedness of the Hardy-Littlewood maximal operator, the singular integral operator, the fractional integral operator, and the commutator of these operators in Morrey spaces are studied by many authors, see [5, 18-20], and the references therein. In [16], Komori and Shirai studied the boundedness of these operators in weighted spaces. Furthermore, in recent years, the study of function spaces associated with different operators inspired great interests, see [1, 4, 8-10, 14, 15]. Meanwhile, the Littlewood-Paley function associated with operator was also studied extensively, see, for example, [2, 3, 6, 11-13, 23]. In this paper, we study some norm inequalities for the area integrals associated to a non-negative self-adjoint operator satisfying a pointwise Gaussian estimate for its heat kernel, on weighted Morrey spaces.

Unless otherwise specified in the sequel, we always assume that L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ and that the semigroup e^{-tL}, generated by L on $L^2(\mathbb{R}^n)$, has the kernel $p_t(x, y)$, which satisfies the following Gaussian upper bound:

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{ct}\right), \tag{GE}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$, where C and c are positive constants.

For $f \in \mathcal{S}(\mathbb{R}^n)$, define the area functions S_P and S_H by

$$S_P f(x) = \left(\int_{|x-y|<t} |t\sqrt{L}e^{-t\sqrt{L}} f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{1.1}$$

$$S_H f(x) = \left(\int_{|x-y|<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}. \tag{1.2}$$

It is well known (cf., e.g., [21]) that when $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, the classical area functions S_P and S_H are all bounded on

$L^p(\mathbb{R}^n)$, $1 < p < \infty$. For a general non-negative self-adjoint operator L, L^p-boundedness of the area functions S_P and S_H associated to L has been studied extensively, see, for example, [2, 3, 6, 11-13], and the references therein.

Let $1 \leq p < \infty$, $0 < \kappa < 1$, and w be a weight. The weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \{f \in L^p_{\text{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^{\kappa}} \int_B |f|^p w dx \right)^{1/p},$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. If $w = 1$ and $\kappa = \lambda / n$ with $0 < \lambda < n$, then $L^{p,\kappa}(w) = L^{p,\lambda}(\mathbb{R}^n)$, the classical Morrey spaces.

The main results of this paper is the following theorem:

Theorem 1.1. *Let L be a non-negative self-adjoint operator, such that the corresponding heat kernels satisfy Gaussian bounds* (GE). *Let $1 < p < \infty$ and $0 < \kappa < 1$. If $w \geq A_p$, then there exists a constant C, such that*

$$\|S_P f\|_{L^{p,\kappa}(w)} \leq C\| f\|_{L^{p,\kappa}(w)}, \tag{1.3}$$

for all $f \in L^{p,\kappa}(w)$. Also, estimate (1.3) *holds for the operator S_H.*

Throughout, the letters "c" and "C" will denote (possibly different) constants that are independent of the essential variables.

2. Notation and Preliminaries

The standard Hardy-Littlewood maximal function $M_r f$, $1 \leq r < \infty$ is defined by

$$M_r f(x) = \sup_{B: x \in B} \left(\frac{1}{|B|} \int_B |f(y)|^r dy \right)^{1/r},$$

where the sup is taken over all balls containing x. If $r = 1$, $M_1 f$ will be denoted by Mf. The Fefferman-Stein sharp maximal function of f, $M^\sharp f(x)$, is defined by

$$M^\sharp f(x) = \sup_{B: x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy,$$

where $f_B = \frac{1}{|B|} \int_B f dx$.

A weight w is a non-negative locally integrable function. We say that $w \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_B w dx \right) \left(\frac{1}{|B|} \int_B w^{1-p'} dx \right)^{p-1} \leq C,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. For $p = 1$, we say that $w \in A_1(\mathbb{R}^n)$, if there is a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|} \int_B w dy \leq C w(x), \quad \text{for a.e. } x \in B,$$

or, equivalently, $M(w) \leq Cw$ a.e.. We denote $A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$.

For the above definition, see [22].

Let us recall that, if L is a self-adjoint positive definite operator acting on $L^2(\mathbb{R}^n)$, then it admits a spectral resolution

$$L = \int_0^\infty \lambda dE(\lambda).$$

For every bounded Borel function $F : [0, \infty) \to \mathbb{C}$, by using the spectral theorem, we can define the operator

$$F(L) := \int_0^\infty F(\lambda) dE_L(\lambda). \tag{2.1}$$

The following results are useful for certain estimates later:

Lemma 2.1. *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *be even,* $\operatorname{supp}\varphi \subset (-1, 1)$. *Let* Φ *denote the Fourier transform of* φ. *Then for every* $\kappa = 0, 1, 2, \dots,$ *and for every* $t > 0$, *the kernel* $K_{(t^2L)^\kappa \Phi(t\sqrt{L})}(x, y)$ *of the operator* $(t^2L)^\kappa \Phi(t\sqrt{L})$, *which was defined by the spectral theory, satisfies*

$$\operatorname{supp} K_{(t^2L)^\kappa \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \le t\}, \tag{2.2}$$

and

$$|K_{(t^2L)^\kappa \Phi(t\sqrt{L})}(x, y)| \le Ct^{-n}, \tag{2.3}$$

for all $t > 0$ *and* $x, y \in \mathbb{R}^n$.

Proof. The proof of this lemma is standard, see [15] and [11]. □

Lemma 2.2. *For* $1 < p < \infty, 0 < \kappa < 1$, *and* $w \in A_p$, *we have* $\|Mf\|_{L^{p,\kappa}(w)} \le C\|f\|_{L^{p,\kappa}(w)}$.

For the proof of this lemma, see [5, Theorem 3.2].

Lemma 2.3. *Let* $w \in A_\infty$, $0 < \kappa < 1$, *and* $1 < p < \infty$. *Then, for every* $f \in L^1_{\text{loc}}$ *with* $Mf \in L^{p,\kappa}(w)$, *there exists a constant* C_w, *which only depend on w such that*

$$\|Mf\|_{L^{p,\kappa}(w)} \leq C_w \|M^\sharp f\|_{L^{p,\kappa}(w)}. \tag{2.4}$$

Proof. It is well known that the following weighted good λ inequality holds:

$$w\{x \in Q : Mf(x) > 2\lambda,\ M^\sharp f(x) < \gamma\lambda\} \leq C\gamma^r w(Q), \tag{2.5}$$

for all $\lambda, \gamma > 0$, some $r > 0$, and for any cube Q, which includes a point x_0 such that $Mf(x_0) \leq \lambda$. For the proof of (2.5), see [7, Lemma 7.10].

Let B be a ball in $\mathbb{R}^n$. Set $E_\lambda = \{x \in B : Mf(x) > \lambda\}$. Then from Whitney decomposition theorem, we know that there exist mutually disjoint cubes Q_k, such that $E_\lambda = \bigcup_k Q_k$ and $10Q_k \bigcap B \setminus E_\lambda \neq \phi$. Denote $\widetilde{Q}_k = 10Q_k$. Then, there exists a $x_k \in \widetilde{Q}_k \bigcap B \setminus E_\lambda$, that is, $Mf(x_k) \leq \lambda$. From (2.5), we have

$$w\{x \in \widetilde{Q}_k : Mf(x) > 2\lambda,\ M^\sharp f(x) \leq \gamma\lambda\} \leq C\gamma^r w(\widetilde{Q}_k).$$

Set $U_\lambda = \{x \in B : Mf(x) > 2\lambda,\ M^\sharp f(x) \leq \gamma\lambda\}$ and so $U_\lambda \subset E_\lambda = \bigcup_k Q_k \subset \bigcup_k \widetilde{Q}_k$. Then,

$$\begin{aligned} w(U_\lambda) &\leq \sum_k w\{x \in \widetilde{Q}_k : Mf(x) > 2\lambda,\ M^\sharp f(x) \leq \gamma\lambda\} \\ &\leq C\gamma^r \sum_k w(\widetilde{Q}_k) \leq C\gamma^r \sum_k w(Q_k) = C\gamma^r w(E_\lambda) \\ &= C\gamma^r w\{x \in B : Mf(x) > \lambda\}, \end{aligned}$$

where we used the fact that A_∞ weights are doubling measures and C is a constant that only depends on the weight. One can prove that

$$\begin{aligned} \int_B |Mf|^p w dx &= 2^p \int_0^\infty p\lambda^{p-1} w\{x \in B : Mf(x) > 2\lambda\} d\lambda \\ &\leq 2^p \int_0^\infty p\lambda^{p-1} \Big(w(U_\lambda) + w\{x \in B : M^\sharp f(x) > \gamma\lambda\}\Big) d\lambda \end{aligned}$$

$$\leq C2^p \gamma^r \int_B |Mf|^p w dx + \frac{2^p}{\gamma^p} \int_B |M^\sharp f|^p w dx.$$

Let us choose γ such that $C2^p \gamma^r = 1/2$. The former inequality turns out to be

$$\int_B |Mf|^p w dx \leq 2 \frac{2^p}{\gamma^p} \int_B |M^\sharp f|^p w dx.$$

This implies that

$$\|Mf\|_{L^{p,\kappa}(w)} \leq C \|M^\sharp f\|_{L^{p,\kappa}(w)}.$$

The proof of this lemma is completed. □

3. An Auxiliary $g^*_{\mu, \Psi}$ Function

Let $\varphi \in C_0^\infty(\mathbb{R})$ be even function with $\int \varphi = 1$, $\operatorname{supp} \varphi \subset (-1/10, 1/10)$. Let Φ denote the Fourier transform of φ and let $\Psi(s) = s^{2n+2}\Phi^3(s)$. We define the $g^*_{\mu, \Psi}$ function by

$$g^*_{\mu, \Psi}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |\Psi(t\sqrt{L}) f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \mu > 1. \tag{3.1}$$

Proposition 3.1. *Let L be a non-negative self-adjoint operator, such that the corresponding heat kernels satisfy condition* (GE). *Then for $f \in \mathcal{S}(\mathbb{R}^n)$, there exists a constant $C = C_{n, \mu, \Psi}$, such that the area integral S_P satisfies the pointwise estimate*

$$S_P f(x) \leq C g^*_{\mu, \Psi}(f)(x). \tag{3.2}$$

Estimate (3.2) *also holds for the area integral S_H.*

For the proof of Proposition 3.1, see [11] and [13].

Note that from Proposition 3.1, the area functions S_H and S_P are all controlled by the $g_{\mu, \Psi}^{*}$ function. In order to prove Theorem 1.1, it suffices to show the following result:

Theorem 3.2. *Let L be a non-negative self-adjoint operator, such that the corresponding heat kernels satisfy Gaussian bounds* (GE). *Let* $\mu > 3$, $1 < p < \infty$, *and* $0 < \kappa < 1$. *If* $w \in A_p$, *then there exists a constant C such that*

$$\|g_{\mu, \Psi}^{*} f\|_{L^{p, \kappa}(w)} \leq C\|f\|_{L^{p, \kappa}(w)}, \tag{3.3}$$

for all $f \in L^{p, \kappa}(w)$.

To prove Theorem 3.2, we need the following result:

Lemma 3.3. *Let L be a non-negative self-adjoint operator, such that the corresponding heat kernels satisfy Gaussian bounds* (GE). *If* $\mu > 3$, *then for any* $\eta > 1$, *there is a constant* $C > 0$ *such that*

$$M^{\sharp}(g_{\mu, \Psi}^{*}(f))(x) \leq CM_{\eta}f(x). \tag{3.4}$$

Proof. Let $T(B) = \{(y, t) : y \in B, 0 < t < r_B\}$, where r_B denotes the radius of B. For $(y, t) \in T(B)$, using (2.2) of Lemma 2.1, we have

$$\Psi(t\sqrt{L})f(y) = \Psi(t\sqrt{L})(f\chi_{3B})(y). \tag{3.5}$$

Let $\mu > 3$ and $\eta > 1$. To prove (3.4), we will show that for each ball B containing x and for some constant c_B, there exists a positive constant C, such that

$$\frac{1}{|B|}\int_B |g_{\mu, \Psi}^{*}(f)(z) - c_B| dz \leq CM_{\eta}f(x).$$

Now, fix a ball B containing x. Denote $\mathbb{R}_{+}^{n} = \mathbb{R}^n \times (0, \infty)$. For any $z \in B$, we decompose $(g_{\mu, \Psi}^{*}(f)(z))^2$ into the sum of

$$I_1(z) = \iint_{T(2B)} |\Psi(t\sqrt{L})f(y)|^2 \left(\frac{t}{t + |z - y|}\right)^{n\mu} \frac{dydt}{t^{n+1}},$$

and

$$I_2(z) = \iint_{\mathbb{R}_+^n \setminus T(2B)} |\Psi(t\sqrt{L})f(y)|^2 \left(\frac{t}{t + |z - y|}\right)^{n\mu} \frac{dydt}{t^{n+1}}.$$

We take $c_B = I_2(z_B)^{1/2}$, where z_B is the center of B. Now, since $\left||a|^s - |b|^s\right| \le |a - b|^s$ for $0 < s < 1$, we have

$$\begin{aligned} |g_{\mu, \Psi}^*(f)(z) - I_2(z_B)^{1/2}| &= |(I_1(z) + I_2(z))^{1/2} - I_2(z_B)^{1/2}| \\ &\le |I_1(z) + I_2(z) - I_2(z_B)|^{1/2} \\ &\le I_1(z)^{1/2} + |I_2(z) - I_2(z_B)|^{1/2}. \end{aligned}$$

This implies

$$\begin{aligned} &\frac{1}{|B|}\int_B |g_{\mu, \Psi}^*(f)(z) - I_2(z_B)^{1/2}| dz \\ &\quad \le \frac{1}{|B|}\int_B I_1(z)^{1/2} dz + \frac{1}{|B|}\int_B |(I_2(z) - I_2(z_B)|^{1/2} dz \\ &\quad =: II_1 + II_2. \end{aligned}$$

It is well known that for $\eta > 1$, $g_{\mu, \Psi}^*(f)$ is bounded on $L^{\eta}(\mathbb{R}^n)$ (see [12]). Then, we have

$$\begin{aligned} \int_B |g_{\mu, \Psi}^*(f\chi_{6B})(z) dz &= \int_0^\infty |\{z \in B : g_{\mu, \Psi}^*(f\chi_{6B})(z) > t\}| dt \\ &\le \int_0^\infty \min\left(C\frac{\|f\chi_{6B}\|_\eta^\eta}{t^\eta}, |B|\right) dt \end{aligned}$$

$$\leq \int_0^{\frac{\|f\chi_{6B}\|_\eta}{|B|^{1/\eta}}} |B| dt + C\int_{\frac{\|f\chi_{6B}\|_\eta}{|B|^{1/\eta}}}^{\infty} \frac{\|f\chi_{6B}\|_\eta^\eta}{t^\eta} dt$$

$$\leq C\|f\chi_{6B}\|_\eta |B|^{1-1/\eta}.$$

This, together with (3.5), implies that

$$II_1 \leq \frac{1}{|B|}\int_B g^*_{\mu,\Psi}(f\chi_{6B})(z)dz$$

$$\leq C\left(\frac{1}{|B|}\int_{6B}|f(z)|^\eta dz\right)^{1/\eta}$$

$$\leq CM_\eta f(x). \tag{3.6}$$

Further, by the mean value theorem, we know that for $z \in B$ and $(y, t) \notin T(2B)$, there exists $0 < s \leq 1$, such that

$$(t + |z - y|)^{-n\mu} - (t + |z_B - y|)^{-n\mu} \leq Cr_B^s(t + |z - y|)^{-n\mu - s}.$$

From this and (3.5), using Lemma 2.1, Hölder's inequality, and $\mu > 3$, we have

$$|I_2(z) - I_2(z_B)|$$

$$\leq Cr_B^s \iint_{\mathbb{R}^n_+\setminus T(2B)} t^{n\mu}|\Psi(t\sqrt{L})f(y)|^2\left(\frac{1}{(t + |z - y|)}\right)^{n\mu + s}\frac{dydt}{t^{n+1}}$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{2^{sk}(2^k r_B)^{n\mu}}\iint_{T(2^{k+1}B)\setminus T(2^k B)}|\Psi(t\sqrt{L})f(y)|^2\frac{dydt}{t^{n+1-n\mu}}$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{2^{sk}(2^k r_B)^{n\mu}}\left(\int_0^{2^{k+1}r_B}\int_{2^{k+1}B}\frac{dydt}{t^{1+3n-n\mu}}\right)\left(\int_{6\cdot 2^k B}|f(y)|dy\right)^2$$

$$\leq C\sum_{k=1}^{\infty}\frac{1}{2^{sk}}\left(\frac{1}{|2^{k+1}B|}\int_{6\cdot 2^k B}|f(y)|dy\right)^2 \leq CM_\eta f(x)^2,$$

for all $z \in B$. Combining this estimate with (3.6) yields

$$\frac{1}{|B|}\int_B |g^*_{\mu,\Psi}(f)(z) - I_2(z_B)^{1/2}|\,dz \le CM_\eta f(x),$$

and then the desired estimate (3.4) holds. This concludes the proof of this lemma. □

The proof of Theorem 3.2. Let $\mu > 3$, $1 < p < \infty$, and $w \in A_p$. Then, there exists $1 < \eta < p$ such that $w \in A_{p/\eta}$. Using Lemmas 2.3 and 3.3, we obtain that

$$\|g^*_{\mu,\Psi}(f)\|_{L^{p,\kappa}(w)} \le \|M^\sharp(g^*_{\mu,\Psi}(f))\|_{L^{p,\kappa}(w)} \le C\|M_\eta(f)\|_{L^{p,\kappa}(w)}.$$

This, combining with Lemma 2.2, gives that

$$\|g^*_{\mu,\Psi}(f)\|_{L^{p,\kappa}(w)} \le C\|M_\eta(f)\|_{L^{p,\kappa}(w)} = C\|M(|f|^\eta)\|^{1/\eta}_{L^{p/\eta,\kappa}(w)}$$

$$\le C\||f|^\eta\|^{1/\eta}_{L^{p/\eta,\kappa}(w)} = C\|f\|_{L^{p,\kappa}(w)}.$$

The proof of Theorem 3.2 is completed. □

Remark. For $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Littlewood-Paley-Stein functions g_P and g_H by

$$g_p(f)(x) = \left(\int_0^\infty |t\sqrt{L}e^{-t\sqrt{L}}f(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

$$g_h(f)(x) = \left(\int_0^\infty |t^2Le^{-t^2L}f(x)|^2\,\frac{dt}{t}\right)^{1/2}.$$

Then, one has the analogous statement as in Theorem 1.1 replacing S_P, S_H by g_P, g_H, respectively.

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